ON THE STABILITY OF PERMANENT ROTATIONS OF A HEAVY GYROSTAT

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Rumiantsev derived in [1] the necessary conditions of stability of rotations of a heavy gyrostat moving by inertia, for several different cases of mass distribution.

The author of this paper derives necessary conditions of stability of permanent rotations of a heavy gyrostat with arbitrary mass distribution and determines the regions of stability on the conical locus formed by the permanent axes.

1. Permanent axes. Let the gyrostat S consist of the rigid body S_1 with one fixed point O and of several other bodies S_2 connected with S_1. Let $O\xi\eta\zeta$ be a fixed rectangular coordinate system with the vertical ζ axis directed upwards, and let $Oxyz$ be a rectangular coordinate system moving with the body S_1, whose axes coincide with the principal axes of inertia of the gyrostat. It should be mentioned that by the definition of a gyrostat as given in [2], the internal movements of the bodies S_2 do not change either the center of gravity, or the orientation of the principal axes, or the moments of inertia of the gyrostat with respect to the point O.

The angular momentum $\mathbf{K}$ of the gyrostat about the point O is the vector sum of the angular momentum $\mathbf{K}_1$ of the whole system S about O, regarded as a rigid body, and of the angular momentum $\mathbf{K}_2$ of the bodies S_2 about O (gyrostatic moment), resulting from the internal movements of S_2 with respect to S_1. Thus

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 \qquad (K_1 = \{Ap,\ Bq,\ Cr\},\ K_2 = \{a, b, c\})$$

The above formulas display the x, y, z, components of the vectors of the considered angular moments. Here A, B, C, are the principal moments of inertia of the gyrostat, and p, q, r, are the x, y, z, components of the angular velocity vector ω. We shall assume that the x, y, z, components of the vector of the gyrostatic moment $\mathbf{K}_2$ are

$$a = \text{const}, \quad b = \text{const}, \quad c = \text{const} \tag{1.1}$$

The motion of a heavy gyrostat with one fixed point under the conditions (1.1) is controlled by the system of six equations

$$\begin{aligned} A\frac{dp}{dt} + (C-B)\,qr + qc - rb &= P\,(z_0\gamma_2 - y_0\gamma_3) \\ B\frac{dq}{dt} + (A-C)\,rp + ra - pc &= P\,(x_0\gamma_3 - z_0\gamma_1) \\ C\frac{dr}{dt} + (B-A)\,pq + pb - qa &= P\,(y_0\gamma_1 - x_0\gamma_2) \end{aligned} \tag{1.2}$$

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \qquad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \qquad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2 \tag{1.3}$$

Here P is the weight of the gyrostat,, x_0, y_0, z_0 are the coordinates of the center of gravity G, and γ_1, γ_2, γ_3, are the direction cosines of the ζ axis with respect to the axes x, y, z. Equations (1.2) and (1.3) permit the three first integrals

$$Ap^2 + Bq^2 + Cr^2 + 2P\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = \text{const} \tag{1.4}$$

$$(Ap + a)\,\gamma_1 + (Bq + b)\gamma_2 + (Cr + c)\,\gamma_3 = \text{const} \tag{1.5}$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{1.6}$$

Reasoning, as in the case of a rigid body with one point fixed [2], we find that a gyrostat under the conditions (1.1) can rotate permanently about a fixed vertical axis (through the space and through the body S_1).

Let α, β, γ, be the direction cosines of a vertical permanent axis with respect to the axes x, y, z. The x, y, z components of the vector ω (ω = const) can be written as

$$p = \omega\alpha, \qquad q = \omega\beta, \qquad r = \omega\gamma \tag{1.7}$$

and x, y, z, components as $\mathbf{K}_1$ as

$$A\omega\alpha, \qquad B\omega\beta, \qquad C\omega\gamma \tag{1.8}$$

If we take into account (1.7) and (1.8), the equations of motion (1.2) assume the form

$$\begin{aligned} (C-B)\,\omega^2\beta\gamma + \omega\,(\beta c - \gamma b) &= P\,(z_0\beta - y_0\gamma) \\ (A-C)\,\omega^2\gamma\alpha + \omega\,(\gamma a - \alpha c) &= P\,(x_0\gamma - z_0\alpha) \\ (B-A)\,\omega^2\alpha\beta + \omega\,(\alpha b - \beta a) &= P\,(y_0\alpha - x_0\beta) \end{aligned} \tag{1.9}$$

and Equations (1.3) become identities.

Equations (1.9) determining the magnitude and the sign of the angular velocity of the permanent rotations are consistent if

$$\frac{\beta c-\gamma b}{(C-B)\beta\gamma}=\frac{\gamma a-\alpha c}{(A-C)\alpha\gamma}=\frac{\alpha b-\beta a}{(B-A)\alpha\beta}=2m \tag{1.10}$$

$$\frac{z_0\beta-y_0\gamma}{(C-B)\beta\gamma}=\frac{x_0\gamma-z_0\alpha}{(A-C)\alpha\gamma}=\frac{y_0\alpha-x_0\beta}{(B-A)\alpha\beta}=\frac{n}{p} \tag{1.11}$$

Dividing (1.10) by (1.11) we obtain

$$\frac{\beta c-\gamma b}{z_0\beta-y_0\gamma}=\frac{\gamma a-\alpha c}{x_0\gamma-z_0\alpha}=\frac{\alpha b-\beta a}{y_0\alpha-x_0\beta}=\frac{2mp}{n} \tag{1.12}$$

Consequently the vectors ω, $\mathbf{K}_2$ and OG must be coplanar, that is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ a & b & c \\ x_0 & y_0 & z_0 \end{vmatrix}=0 \tag{1.13}$$

We shall assume in what follows that the condition (1.13) is satisfied.

Equations (1.9) can be written in the form

$$\omega^2+2m\omega-n=0 \tag{1.14}$$

Multiplying Equations (1.9) by x_0, y_0, z_0, respectively, adding them and taking into account (1.13) we obtain the equations

$$(C-B)x_0\beta\gamma+(A-C)y_0\gamma\alpha+(B-A)z_0\alpha\beta=0 \tag{1.15}$$

representing the locus of the vertical permanent axes.

The locus of Equation (1.15) is a second order cone in the variables α, β, γ. In the cone (1.15) the principal axes of inertia x, y, z, and the lines through the cone's vertex O and through the points $G(x_0, y_0, z_0)$ and $F(x_0/A, y_0/B, z_0/C)$ are generatrices. These five lines determine the cone completely.

The intersection of a unit sphere centered on O with the generatrices of the cone consists of two closed branches of a spherical curve. Let x, y, z, g, f, be the points of intersection of the generatrices Ox, Oy, Oz, OG, OF, with the unit sphere, and let $-x$, $-y$, $-z$, $-g$, $-f$ be the points diametrically opposite. One branch of the spherical curve is through the points x, g, f, z, $-y$, and the other one through the points $-x$, $-g$, $-f$, $-z$, y.

A semigeneratrix of the cone (1.15) determined by α, β, γ, which determine in turn the angular velocity ω in (1.14), can make a permanent axis, and will be called "the admissible semigeneratrix".

We shall assume that

$$A > B > C, \qquad x_0 > 0, \qquad y_0 > 0, \qquad z_0 > 0 \tag{1.16}$$

$$\alpha \neq 0, \qquad \beta \neq 0, \qquad \gamma \neq 0 \tag{1.17}$$

For the semigeneratrices passing through the points $(x, -y)$, (z, g) of the first branch, and through the points $(y, -z)$ and $(-g, -x)$ of the second branch of the spherical curve we have $n > 0$; consequently, these semigeneratrices are admissible irrespective of the value of m, which is determined from (1.12). The semigeneratrices through $(-y, z)$, (g, x), $(-z, -g)$, and $(-x, y)$, when $m = 0$, cannot be permanent axes because $n < 0$; when $m \neq 0$ these semigeneratrices may become admissible if

$$m^2 + n > 0$$

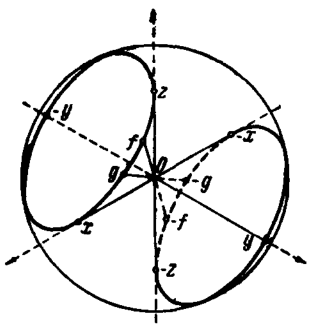

Fig. 1.

We shall consider the case when the permanent axis coincides with one of the principal axes of inertia, for example with the x axis. In this case

$$\alpha = 1, \qquad \beta = \gamma = 0 \tag{1.18}$$

From (1.13) follows that

$$\frac{b}{y_0} = \frac{c}{z_0} \tag{1.19}$$

and from (1.9) we find the angular velocity of the permanent rotation

$$\omega = \frac{Pz_0}{c} = \frac{Py_0}{b} \tag{1.20}$$

It should be mentioned that in general

$$\frac{x_0}{a} \neq \frac{y_0}{b} = \frac{z_0}{c} \tag{1.21}$$

If the gyrostatic moment is colinear with the vector OG, that is when

$$\frac{x_0}{a} = \frac{y_0}{b} = \frac{z_0}{c} \tag{1.22}$$

then the permanent rotations about each of the principal axes of inertia has the same angular velocity

$$\omega = \frac{Px_0}{a} = \frac{Py_0}{b} = \frac{Pz_0}{c} \tag{1.23}$$

2. Stability of permanent rotations. Let α, β, γ, under the conditions (1.17) determine any admissible semigeneratrix of the cone (1.15) which can be a permanent (vertical) axis of rotation of the gyrostat. The x, y, z, components of the constant angular velocity

$$p_0 = \alpha\omega, \qquad q_0 = \beta\omega, \qquad r_0 = \gamma\omega \tag{2.1}$$

are found from (1.14).

We shall investigate the stability of permanent rotations of the gyrostat with respect to the variables p, q, r, γ_1, γ_2, γ_3. Substituting in Equations (1.2) and (1.3)

$$\begin{aligned} p &= p_0 + \xi_1, \qquad q = q_0 + \xi_2, \qquad r = r_0 + \xi_3 \\ \gamma_1 &= \alpha + \eta_1, \qquad \gamma_2 = \beta + \eta_2, \qquad \gamma_3 = \gamma + \eta_3 \end{aligned} \tag{2.2}$$

and taking into account (1.9) and (2.1), we obtain the equations of the perturbed motion of the gyrostat, which permit the following first integrals

$$\begin{aligned} V_1 &= A\,(\xi_1^2 + 2p_0\xi_1) + B\,(\xi_2^2 + 2q_0\xi_2) + C\,(\xi_3^2 + 2r_0\xi_3) + \\ &\quad + 2P\,(x_0\eta_1 + y_0\eta_2 + z_0\eta_3) = \text{const} \\ V_2 &= A\,(p_0\eta_1 + \alpha\xi_1 + \xi_1\eta_1) + B\,(q_0\eta_2 + \beta\xi_2 + \xi_2\eta_2) + \\ &\quad + C\,(r_0\eta_3 + \gamma\xi_3 + \xi_3\eta_3) + a\eta_1 + b\eta_2 + c\eta_3 = \text{const} \\ V_3 &= \eta_1^2 + \eta_2^2 + \eta_3^2 + 2\,(\alpha\eta_1 + \beta\eta_2 + \gamma\eta_3) = 0 \end{aligned} \tag{2.3}$$

Let us construct the Liapunov function in the form

$$\begin{aligned} V = V_1 - 2\omega V_2 + \lambda V_3 &= A\xi_1^2 + B\xi_2^2 + C\xi_3^2 - 2A\omega\xi_1\eta_1 - \\ &- 2B\omega\xi_2\eta_2 - 2C\omega\xi_3\eta_3 + \lambda\eta_1^2 + \lambda\eta_2^2 + \lambda\eta_3^2 \end{aligned} \tag{2.4}$$

where by (1.9) the constant λ equals

$$\lambda = A\omega^2 + \frac{a\omega - Px_0}{\alpha} = B\omega^2 + \frac{b\omega - Py_0}{\beta} = C\omega^2 + \frac{c\omega - Pz_0}{\gamma} \tag{2.5}$$

The necessary and sufficient conditions for positive-definiteness of the function V are by Sylvester's criterion

$$\begin{gathered} A > 0, \qquad AB > 0, \qquad ABC > 0, \qquad \lambda - A\omega^2 > 0 \\ (\lambda - A\omega^2)\,(\lambda - B\omega^2) > 0, \qquad (\lambda - A\omega^2)\,(\lambda - B\omega^2)\,(\lambda - C\omega^2) > 0 \end{gathered} \tag{2.6}$$

The first three inequalities are always satisfied, and the remaining three are satisfied if

$$\lambda - A\omega^2 > 0, \qquad \lambda - B\omega^2 > 0, \qquad \lambda - C\omega^2 > 0 \tag{2.7}$$

Substituting the value of λ from (2.5) into (2.7) we find

$$\frac{a\omega - Px_0}{\alpha} > 0, \qquad \frac{b\omega - Py_0}{\beta} > 0, \qquad \frac{c\omega - Pz_0}{\gamma} > 0 \tag{2.8}$$

The function V under the conditions (2.8) is a sign-definite integral of the perturbed motion of the gyrostat, consequently the conditions (2.8) are, by Liapunov's theorem, the necessary conditions of stability of the permanent rotations of the gyrostat.

If the gyrostatic moment is collinear with the vector ω, that is

$$\frac{a}{\alpha} = \frac{b}{\beta} = \frac{c}{\gamma} = \mu\omega \tag{2.9}$$

then by (1.10) we have $m = 0$. In this case the admissible permanent axes are only those semigeneratrices of the cone (1.15) which are determined by the arcs $(x, -y)$, (z, g), $(y, -z)$ and $(-g, -x)$. Under the conditions (2.9) the sufficient conditions of stability (2.8) can be written in the form

$$\mu - \frac{Px_0}{\alpha\omega^2} > 0, \qquad \mu - \frac{Py_0}{\beta\omega^2} > 0, \qquad \mu - \frac{Pz_0}{\gamma\omega^2} > 0 \tag{2.10}$$

If $\mu > 0$, then from (2.10) follows that the permanent rotations of a gyrostat are stable for every axis coinciding with a semigeneratrix of the cone which passes through the points of the arc $(-g, -x)$. For permanent axes determined by other admissible arcs, the sufficient conditions (2.10) are satisfied if μ is sufficiently large. For example, for the arc $(x, -y)$ the sufficient conditions of stability are

$$\mu > \frac{Px_0}{\alpha\omega^2} \tag{2.11}$$

If $\mu < 0$, then the sufficient conditions can be satisfied only for the points of the arc $(-g, -x)$ when

$$|\mu| < \left|\frac{Px_0}{\alpha\omega^2}\right|, \qquad |\mu| < \left|\frac{Py_0}{\beta\omega^2}\right|, \qquad |\mu| < \left|\frac{Pz_0}{\gamma\omega^2}\right| \tag{2.12}$$

If a permanent axis passes through the center of gravity, that is

$$\frac{x_0}{\alpha} = \frac{y_0}{\beta} = \frac{z_0}{\gamma} \tag{2.13}$$

then by (1.11) we have $n = 0$, and for permanent rotations about this axis we obtain from (1.14) $\omega_1 = 0$ and $\omega_2 = -2m$.

For permanent axes coinciding with the semigeneratrix through g, we have $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and the sufficient conditions of stability are

$$a\omega - Px_G > 0, \qquad b\omega - Py_0 > 0, \qquad c\omega - Pz_0 > 0 \qquad (2.14)$$

which cannot be satisfied when $\omega = 0$.

For permanent axes determined by $-$ g, we have $\alpha < 0$, $\beta < 0$, $\gamma < 0$ and the sufficient conditions of stability obtained from (2.8) are

$$a\omega - Px_0 < 0, \qquad b\omega - Py_0 < 0, \qquad c\omega - Pz_0 < 0 \qquad (2.15)$$

The inequalities (2.15) show clearly that in this case the equilibrium of the gyrostat is stable.

3. Stability of permanent rotations of a gyrostat about any of its principal axes of inertia. Let the permanent axis of rotation be the x-axis. The particular solution for this case is

$$p_0 = \omega = \frac{Py_0}{b} = \frac{Pz_0}{c}, \quad q_0 = 0, \quad r_0 = 0, \quad \alpha = 1, \quad \beta = 0, \quad \gamma = 0 \qquad (3.1)$$

Substituting in Equations (1.2) and (1.3)

$$p = \omega + \xi_1, \quad q = \xi_2, \quad r = \xi_3, \quad \gamma_1 = 1 + \eta_1, \quad \gamma_2 = \eta_2, \gamma_3 = \eta_3 \quad (3.2)$$

and taking into account (1.9) and (3.1), we obtain the equations of the perturbed motion, which admit the following first integrals

$$V_1 = A(\xi_1^2 + 2\omega\xi_1) + B\xi_2^2 + C\xi_3^2 + 2P(x_0\eta_1 + y_0\eta_2 + z_0\eta_3) = \text{const}$$
$$V_2 = A(\omega\eta_1 + \xi_1 + \xi_1\eta_1) + B\xi_2\eta_2 + C\xi_3\eta_3 + a\eta_1 + b\eta_2 + c\eta_3 = \text{const}$$
$$V_3 = \eta_1^2 + \eta_2^2 + \eta_3^2 + 2\eta_1 = 0 \qquad (3.3)$$

We shall construct the Liapunov function in the form

$$V = V_1 - 2\omega V_2 + \lambda V_3 = A\xi_1^2 + B\xi_2^2 + C\xi_3^2 - 2A\omega\xi_1\eta_1 -$$
$$- 2B\omega\xi_2\eta_2 - 2C\omega\xi_3\eta_3 + \lambda\eta_1^2 + \lambda\eta_2^2 + \lambda\eta_3^2 \qquad (3.4)$$

where the constant λ equals

$$\lambda = A\omega^2 + a\omega - Px_0 \qquad (3.5)$$

The necessary and sufficient conditions for the positive-definiteness of the function V are the inequalities (2.7), which are therefore the sufficient conditions of stability of permanent rotations about the x-axis. Let

$$A > B, \qquad A > C \qquad (3.6)$$

Taking into account (3.5) and (3.6) we have the sufficient conditions

If
$$a\omega - Px_0 > 0 \tag{3.7}$$
$$B > A > C \quad \text{or} \quad B > C > A \tag{3.8}$$

then the sufficient conditions of stability have the form

$$a\omega - Px_0 > (B - A)\,\omega^2 \tag{3.9}$$

We have obtained in the previous articles the sufficient conditions of stability of permanent rotations about a principal axis of inertia of a heavy gyrostat with one point fixed, in the case when projections of the gyrostatic moment and of the radius vector of the center of gravity on two of the axes are proportional to each other. These conditions are not sufficient when the vector of the gyrostatic moment passes through the center of gravity*.

4. If the vector of the gyrostatic moment passes through the center of gravity of a gyrostat, then [6] the permanent rotations about each principal axis of inertia have the same angular velocity.

We shall consider the permanent rotations of a gyrostat about the x-axis which are determined by the following particular solutions of the equations of motion (1.2) and (1.3)

$$p_0 = \omega = \frac{Px_0}{a} = \frac{Py_0}{b} = \frac{Pz_0}{c}, \qquad q_0 = 0, \quad r_0 = 0$$
$$\gamma_{01} = 1, \qquad \gamma_{02} = 0, \qquad \gamma_{03} = 0 \tag{4.1}$$

Substituting in Equation (1.2) and (1.3)

$$p = \omega + \xi_1, \quad q = \xi_2, \quad r = \xi_3, \quad \gamma_1 = 1 + \eta_1, \quad \gamma_2 = \eta_2, \quad \gamma_3 = \eta_3$$

and taking into account (4.1), we obtain the equation of the perturbed motion of the gyrostat, which permit the following first integrals

$$V_1 = A(\xi_1^2 + 2\omega\xi_1) + B\xi_2^2 + C\xi_3^2 + 2P(x_0\eta_1 + y_0\eta_2 + z_0\eta_3) = \text{const}$$
$$V_2 = A(\omega\eta_1 + \xi_1 + \xi_1\eta_1) + B\xi_2\eta_2 + C\xi_3\eta_3 + a\eta_1 + b\eta_2 + c\eta_3 = \text{const} \tag{4.2}$$
$$V_3 = \eta_1^2 + \eta_2^2 + \eta_3^2 + 2\eta_1 = 0$$

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* Article 4 has been added after the proof-reading, on 18 October 1961.

We shall construct the Liapunov function in the form

$$V = V_1 - 2\omega V_2 + A\omega^2 V_3 + \tfrac{1}{4} V_3^2 = A\xi_1^2 + B\xi_2^2 + C\xi_3^2 - 2A\omega\xi_1\eta_1 -$$
$$- 2B\omega\xi_2\eta_2 - 2C\omega\xi_3\eta_3 + (A\omega^2+1)\eta_1^2 + A\omega^2\eta_2^2 + A\omega^2\eta_3^2 + \tfrac{1}{4} f(\eta_1, \eta_2, \eta_3) \quad (4.3)$$

where

$$f(\eta_1, \eta_2, \eta_3) = (\eta_1^2 + \eta_2^2 + \eta_3^2)(\eta_1^2 + \eta_2^2 + \eta_3^2 + 4\eta_1)$$

The function V is a positive-definite function of the variables ξ_1, ξ_2, ξ_3, η_1, η_2, η_3, if such is its quadratic part. By Sylvester's criterion the necessary and sufficient conditions for positive sign-definiteness of the quadratic part are the inequalities

$$A - B > 0, \qquad (A - B)(A - C) > 0 \qquad (4.4)$$

If the x-axis is the axis of the greatest moment of inertia, then the function V is a sign-definite integral of the equations of the perturbed motion of the heavy gyrostat, and, on the strength of Liapunov's stability theorem, the permanent rotations about the x-axis will be stable.

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